YDK 517.9

QUASILINEAR PARABOLIC BOUNDARY VALUE PROBLEMS

©F.Nicolosi, Department of Mathematics, Univ. of Catania, Italy

We study boundary value problems for second order degenerated quasilinear parabolic equations of divergence form and we establish the existence of the weak solutions which are in L^∞. We consider the equation of the type

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) = f(x,t,u,\nabla u) + u_t \qquad (0.1)$$

$(x,t) \in Q = \Omega \times]0,T[$, where Ω is a bounded open subset of R^m, T a positive real number. We assume that f has a quadratic growth with respect to the gradient ∇u. The left hand side satisfies the ellipticity condition of the type

$$\lambda(|u|) \sum_{i=1}^{m} a_i(x,t,u,p)p_i \geq \nu(x)\,\psi(t)\,|p|^2, \qquad (0.2)$$

where p is the vector $(p_1, p_2, ..., p_m)$, the symbol $|p|$ is the modulos of p, ν and ψ are positive functions, ν is integrable in Ω, ψ is monotone and nondecreasing in $]0,T[$, ν^{-1} is not necessarily bounded and ψ, ψ^{-1} are not necessarily integrable.

Analogous results were obtained by BOCCARDO, MURAT, PUEL, Existence results for some parabolic equations, *Nonlinear Analysis, 13 (1989)*, and by MOKRANE, Existence of bounded solutions of some nonlinear parabolic equations, *Proceedings of the Royal Society of Edinbergh, 107 A(1987)*, in the case of Cauchy-Diriclet problem for nondegenerate equations. However, the degenerate problem is more complicated and the principal difficulty is caused by the presence of a function ψ in the right hand side of (1).

Function spaces, hypotheses and problems.

We shall suppose that R^m $(m \geq 2)$ is m- dimensional euclidean space with elements $x = (x_1, x_2, ..., x_m)$. Let Ω be a bounded open nonempty subset of R^m. Let $0 \leq \tau_1 < \tau_2 \leq T < \infty$. We denote by $Q(\tau_1, \tau_2)$ the cylinder $\Omega \times]\tau_1, \tau_2[$; $Q = Q(0,T)$.

Hypothesis 1. Let $\nu(x)$ be a positive function defined in Ω:

$$\nu(x) \in L^1_{loc}(\Omega), \;\; \nu^{-1}(x) \in L^1_{loc}(\Omega).$$

We shall denote by $H^1(\nu,\Omega)$ the function space of all real functions $u \in L^2(\Omega)$ such that their derivatives (in the sense of distribution in Ω) are functions $\frac{\partial u}{\partial x_i}$ $(i = 1, 2, ..., m)$ which have the following property

$$\sqrt{\nu}\,\frac{\partial u}{\partial x_i} \in L^2(\Omega).$$

Then $H^1(\nu,\Omega)$ is a HILBERT space with the norm

$$||u||_1 = \left(\int_\Omega (|u|^2 + \nu\,|\nabla u|^2)\,dx\right)^{1/2}$$

The closure of C_0^∞ in $H^1(\nu,\Omega)$ will be denoted by $H_0^1(\nu,\Omega)$.

Let V be an arbitrary closed vectorial subspace of $H^1(\nu,\Omega)$ which contains $H_0^1(\nu,\Omega)$, i.e.

$$H_0^1(\nu,\Omega) \subseteq V \subseteq H^1(\nu,\Omega).$$

Let V' be the dual space of V.

Hypothesis (1.2) Let $\psi(t)$ be a function which is positive and nondecreasing in $]0,T[$.
There exists a positive number g such that $\psi^{-1} \in L^g(0,T)$.
(The number g can be also less than one).
We shall denote by $H^{1,0}(\nu\,\psi,Q)$ the function space of all real functions $u \in L^2(Q)$ such that their derivatives (in the sense of distibutions in Q) are functions $\frac{\partial u}{\partial x_i}$ $(i = 1,2,...,m)$ which have the following property

$$\sqrt{\nu\,\psi}\,\frac{\partial u}{\partial x_i} \in L^2(Q).$$

Then $H^{1,0}(\nu\,\psi,Q)$ is a HILBERT space with the norm

$$\|u\|_{1,0} = \Big(\int_Q (|u|^2 + \nu\,\psi\,|\nabla\, u|^2)\,dx\,dt\Big)^{1/2}.$$

We shall denote by $H^{1,1}(\nu\,\psi,Q)$ the function space of all functions $u \in H^{1,0}(\nu\,\psi,Q)$ such that their derivative (in the sense of distributions in Q) $u_t \in L^2(Q)$.

Without the loss of generality we can suppose that any function in $H^{1,1}(\nu\,\psi,Q)$ is continuous in $[0,T]$ as a function with values in $L^2(\Omega)$.

$V^{1,0}(\nu\,\psi,Q)$ denotes the subspace of $H^{1,0}(\nu\,\psi,Q)$ which consists of all functions u such that for a.e. fixed $t \in]0,T[$, $u(x,t)$ belongs to V.

$V^{1,1}(\nu\,\psi,Q) = H^{1,1}(\nu\,\psi,Q) \cap V^{1,0}(\nu\,\psi,Q)$; $V_T^{1,1}(\nu\,\psi,Q)$ denotes the set of all functions of $V^{1,1}(\nu\,\psi,Q)$ such that $u(x,T) = 0$ a.e. in Ω.

$V_T^{1,1}(\nu\,\psi,Q)$ is a HILBERT space equipped with the norm

$$\|u\|_{1,1} = \Big(\int_\Omega (|u|^2 + \nu\,\psi|\nabla u|^2 + |u_t|^2)\,dx\,dt\Big)^{1/2}.$$

Hypothesis (1.3) There exists a positive number k_0 such that for any $u \in V$ it is also $\min(u(x),k) \in V$ for any $k \geq k_0$.

Hypothesis (1.4) For any $u \in V \cap L^\infty(\Omega)$ and for any $\gamma > 0$ it is also

$$u(x)\,|u(x)|^\gamma \in V.$$

Hypothesis (1.5) There are α,β such that $\alpha \in]2,+\infty[$, $\beta \in]0,+\infty[$ and

$$|u|_\alpha \leq \beta\,\|u\|_{1,0}$$

for any $u \in V$, where $|\cdot|_\alpha$ denotes the norm in $L^\alpha(\Omega)$.

Remark 1. Put $V = H_0^1(\nu,\Omega)$ or $H^1(\nu,\Omega)$. Then hypotheses (1.3) and (1.4) are satisfied. Let $\nu^{-1} \in L^\gamma(\Omega)$ with $\gamma > \frac{m}{2}$. Then it follows from Sobolev imbedding theorem that Hypothesis (1.5) is satisfied for $V = H_0^1(\nu,\Omega)$ and also for $V = H^1(\nu,\Omega)$ if the set Ω has cone property.

Hypothesis (1.6) The function $f(x,t,u,p)$, $a_i(x,t,u,p)$ $(i = 1,...,m)$ are Caratheodory's function in $Q \times R^{m+1}$, i.e. measurable with respect to (x,t) for any $(u,p) \in R^{m+1}$ and continuous with respect to (u,p) for almost every $(x,t) \in Q$.
$\lambda : [0,+\infty[\rightarrow [1,+\infty[$ is monotone and nondecreasing.

Hypothesis (1.7) There exists a function $f^*(x,t) \in L^1(Q)$ such that

$$|f(x,t,u,p)| \leq \lambda(|u|)\,[f^*(x,t) + \nu\,\psi\,|p|^2]$$

holds for almost every $(x,t) \in Q$ and for all real numbers $u, p_1, ..., p_m$.

Hypothesis (1.8) There exist nonnegative numbers c_1 and c_2 such that for almost all $(x,t) \in Q$ and for all real numbers $u, p_1, ..., p_m$ the inequality

$$u\, f(x,t,u,p) + c_1 u^2 + \lambda(|u|)\, \nu\, \psi\, |p|^2 + c_2 \geq 0. \tag{1.1}$$

holds.

Hypothesis (1.9) There exists a function $a^*(x,t) \in L^2(Q)$ such that for almost every $(x,t) \in Q$ and for all real numbers $u, p_1, ..., p_m$ the inequality

$$\frac{|a_i(x,t,u,p)|}{\sqrt{\nu(x)\,\psi(t)}} \leq \lambda(|u|)\,[a^*(x,t) + \nu\,\psi\,|p|^2] \tag{1.2}$$

holds.

Hypothesis (1.10) The condition (0.2) is satisfied for almost every $(x,t) \in Q$ and for all real numbers $u, p_1, ..., p_m$.

Hypothesis (1.11) For almost every $(x,t) \in Q$ and for any real numbers $u, p_1, ..., p_m, q_1, ..., q_m$ the inequality

$$\sum_{i=1}^{m} [a_i(x,t,u,p) - a_i(x,t,u,q)](p_i - q_i) \geq 0 \tag{1.3}$$

holds, while the equality holds if and only if $p = q$.

Let all Hypotheses (1.1), (1.2), (1.6), (1.7), (1.9) be satisfied.

Let c_0 be a positive constant.

We formulate the following

Problem (1) Find a function $u(x,t) \in V^{1,0}(\nu\,\psi, Q) \cap L^\infty(Q)$ such that the relation

$$\int_Q \{\sum_{i=1}^{m} a_i(x,t,u,\nabla u)\,\frac{\partial W}{\partial x_i} + c_0\, u\, W + f(x,t,u,\nabla u)\, W - u\, W_t\}\, dx\, dt = 0$$

holds for any $W \in V_T^{1,1}(\nu\,\psi, Q) \cap L^\infty(Q)$.

Problem (2) Find a function $u(x,t) \in V^{1,0}(\nu\,\psi, Q) \cap L^\infty(Q)$ such that the relation

$$\int_Q \{\sum_{i=1}^{m} a_i(x,t,u,\nabla u)\,\frac{\partial W}{\partial x_i} + f(x,t,u,\nabla u)\, W - u\, W_t\}\, dx\, dt = 0$$

holds for any $W \in V_T^{1,1}(\nu\,\psi, Q) \cap L^\infty(Q)$.

Preliminary Lemmas.

Lemma (2.1) Under our hypotheses, let $u \in H^{1,0}(\nu\,\psi, Q)$ and $\{u_n\}$ be a sequence in $H^{1,0}(\nu\,\psi, Q)$ such that there exists a constant k_1 for which $||u_n||_{1,0} \leq$ $\leq k_1$ e $\lambda(|u_n(x,t)|) \leq k_1$ for almost all $(x,t) \in Q$ and for any $n = 1, 2, ...$ For fixed $0 \leq \tau_1 < \tau_2 \leq T$ let us suppose

$$\lim_{n\to\infty} \int_{Q(\tau_1,\tau_2)} |u_n(x,t) - u(x,t)|^2\, dx\, dt = 0,$$

and

$$\lim_{n\to\infty} \int_{Q(\tau_1,\tau_2)} \{\sum_{i=1}^{m} [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)]\,\frac{\partial(u_n - u)}{\partial x_i}\}\, dx\, dt = 0 \tag{2.1}$$

Then

$$\lim_{n\to\infty} \int_{Q(\tau_1,\tau_2)} \nu(x)\,\psi(t)\,|\nabla u_n(x,t) - \nabla u(x,t)|^2\, dx\, dt = 0.$$

Proof of the existence of the solution of problem (1).

The main idea is based on the metod of appriori estimates, the elliptic regularizzation and Leray-Lions theorem.

Hypothesis (2.12) Let the function $\lambda(s) = \lambda$ is a constant. Let us suppose, moreover that for any real numbers $u, p_1, ..., p_m$ and for almost all $(x,t) \in Q$ it is $|f(x,t,u,p)| \le \lambda$.

Let all hypotheses (1.1), (1.2), (1.6), (1.9), (2.12) be satisfied. Let ϵ be a positive number. We formulate the auxiliary elliptic problem:

Problem (3) Find a function $u(x,t) \in V_T^{1,1}(\nu\,\psi, Q)$ such that the relation

$$\int_Q \{\sum_{i=1}^{m} a_i(x,t,u,\nabla u)\frac{\partial W}{\partial x_i} + c_0\, u\, W + f(x,t,u,\nabla u)\, W -$$

$$- u\, W_t + \epsilon\, u_t\, W_t\}\, dx\, dt = 0 \quad (2.2)$$

holds for any $W \in V_T^{1,1}(\nu\,\psi, Q$.

Applying Leray-Lions theorem we prove that the problem (3) has at least one solution u_ϵ. We prove that every solution u_ϵ of the problem (3) satisfies the estimate

$$c_0 \operatorname{esssup} |u_\epsilon| \le \lambda. \quad (2.3)$$

We use the truncation method due to DE GIORGI and lemma of STAMPACCHIA.

We prove also that every solution u_ϵ of the problem (3) satisfies

$$||u_\epsilon||_{1,0} + (\int_Q \epsilon\, |\frac{\partial u_\epsilon}{\partial t}|^2\, dx\, dt)^{1/2} \le k_2 \quad (2.4)$$

where k_2 depends on λ, c_0, T and meas Ω.

In order to prove the existence of the solution of our problems (1) for bounded f we need.

Lemma (2.2) Under our hypotheses, let u be a solution of auxiliary problem (3), then the norm of u_t in $L^2(0,\tau;\, V')$ can be estimated for any fixed positive number $\tau < T$ by a constant dependent only from $m, \lambda, c_0, \tau, T, \int_Q |a^*(x,t)|^2\, dx\, dt$, meas Ω and from $\psi(t)$.

Now it is possible to prove the existence of the solution of the problem (1) with f bounded.

To be precise, under our hypotheses the problem (1) has at least one solution.

A sketch of the proof.

Let $\{\epsilon_n\}$ be a sequence of positive numbers approaching zero. For any positive integer n, we denote u_n a function which belongs to $V_T^{1,1}(\nu\,\psi, Q)$ such that

$$\int_Q \{\sum_{i=1}^{m} a_i(x,t,u_n,\nabla u_n)\frac{\partial W}{\partial x_i} + c_0\, u_n\, W + f(x,t,u_n,\nabla u_n)\, W -$$

$$- u_n\, W_t + \epsilon_n \frac{\partial u_n}{\partial t} W_t\}\, dx\, dt = 0 \quad (2.5)$$

for any $W \in V_T^{1,1}(\nu\,\psi, Q)$. The existence of u_n is due to our presedent results.

It follows from (2.3), (2.4)

$$c_0 \operatorname{esssup}|u_n| \le \lambda, \quad ||u_n||_{1,0} + (\int_Q \epsilon_n\, |\frac{\partial u_n}{\partial t}|^2\, dx\, dt)^{1/2} \le k_2.$$

It is possible substract a subsequence of $\{u_n\}$ (denoted again by $\{u_n\}$) which converges weakly to u in $V^{1,0}(\nu\,\psi, Q)$ and weakly* in $L^\infty(Q)$ to $u \in V^{1,0}(\nu\,\psi, Q) \cap L^\infty(Q)$.

We shall prove that this element $u \in V^{1,0}(\nu\,\psi, Q) \cap L^\infty(Q)$ is the solution of problem (1).

In order to pass to limit in (2.5) for $n \to +\infty$ we need the following auxiliary assertions.

Assertion 1.

$$\lim_{n\to\infty} \int_Q \epsilon_n \frac{\partial u_n}{\partial t}\, w_t\, dx\, dt = 0$$

Assertion 2. The sequence $\{u_n\}$ converges strongly in $L^2(Q)$.

Assertion 3.

$$\lim_{n\to\infty} \int_{Q(\tau_1,\tau_2)} \nu(x)\,\psi(t)\,|\nabla u_n(x,t) - \nabla u(x,t)|^2\, dx\, dt = 0.$$

for fixed $0 < \tau_1 < \tau_2 < T$.

Let us suppose for a moment that the assertion 1-3 are true. From assertion 2-3 it follows that $\{u_n\}$ has a subsequence $\{u_{Z_n}\}$ such that $\{u_{Z_n}\}$ and $\{\nabla u_{Z_n}\}$ converge a.e. in Q. Let us write (2.5) for $\{u_{Z_n}\}$. Then we can pass to limit in (2.5) for $n \to +\infty$, we obtain

$$\int_Q \{\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial W}{\partial x_i} + c_0\, u\, W + f(x,t,u,\nabla u)\, W - u\, W_t\}\, dx\, dt = 0$$

holds for any $W \in V_T^{1,1}(\nu\,\psi, Q) \cap L^\infty(Q)$, i.e. $u \in V^{1,0}(\nu\,\psi, Q) \cap L^\infty(Q)$ is the solution of the problem (1).

Hence it remains to prove the assertions 1-3.

Proof of assertion 1

$$|\int_Q \epsilon_n \frac{\partial u_n}{\partial t}\, W_t\, dx\, dt| \le \sqrt{\epsilon_n}\, k_2 (\int_Q |W_t|^2\, dx\, dt)^{1/2}$$

hence

$$\lim_{n\to\infty} \int_Q \epsilon_n \frac{\partial u_n}{\partial t}\, W_t\, dx\, dt = 0.$$

Proof of assertion 2

For arbitrary fixed positive number $\tau < T$ the norm of u_n in $L^2(\tau, T;\, V)$ is bounded by $(\max(1, [\psi(\tau)]^{-1}, k_2))^{1/2}$; due to lemma (2.2) the norm of the derivative of u_n with respect to t in $L^2(\tau, T;\, V')$ is bounded by a constant which depends on τ but does not depend on n. Using the results of SIMON, Compact sets in the space $L^p(0,T;\, B)$, *Annali di Matematica, (14) 176 (1987),* $\{u_n\}$ converges to u in $L^2(Q)$; we also that imbedding of V in $L^2(Q)$ is compact, see GUGLIELMINO - NICOLOSI, Sulle W - soluzioni dei problemi al contorno per operatori degeneri, *Ricerche di Matematica, 36 (1987).*

Proof of assertion 3

It is sufficient that the assumption of lemma (2.1) are satisfied. To this aim it is sufficient to verify that for fixed $0 < \tau_1 < \tau_2 < T$ the relation (2.1) holds.

Let us denote ω the limsup of the sequence

$$\int_{Q(\tau_1,\tau_2)} \{\sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)]\, \frac{\partial (u_n - u)}{\partial x_i}\}\, dx\, dt$$

which is composed of nonnegative numbers. Then it is sufficient to prove that $\omega \le 0$.

Finally we prove the existence of the solution of problem (1) with f not bounded and our hypotheses are satisfied with $c_0 > c_1$. Then we can prove an a priori estimate for any solution of problem (1).

More precisely: $\operatorname*{ess\,sup}_Q |u| \leq L$, where $L = (\frac{c_1}{c_0 - c_1})^{1/2}$.

Put in $Q \times R^{m+1}$, $i = 1, 2, \ldots, m$

$$b_i(x,t,u,p) = \begin{cases} a_i(x,t,-L,p) & \text{for } u < -L \\ a_i(x,t,u,p) & \text{for } |u| \leq L \\ a_i(x,t,L,p) & \text{for } u > L \end{cases}$$

and for any positive integer n

$$f_n(x,t,u,p) = \begin{cases} f(x,t,u,p) & \text{for } |f| \leq n \\ \frac{n\, f(x,t,u,p)}{|f(x,t,u,p)|} & |f| > n \end{cases}$$

It follows from our results that for any u there exists a function $u_n(x,t) \in V^{1,0}(\nu\psi, Q) \cap L^\infty(Q)$ which satisfies the relation:

$$\int_Q \{\sum_{i=1}^m b_i(x,t,u_n,\nabla u_n) \frac{\partial W}{\partial x_i} + c_0\, u_n\, W + f_n(x,t,u_n,\nabla u_n)\, W - u_n\, W_t\}\, dx\, dt = 0 \tag{2.6}$$

for any $W \in V_T^{1,1}(\nu\,\psi, Q) \cap L^\infty(Q)$.

An a priori estimate yelds

$$\operatorname*{ess\,sup}_Q |u_n| \leq L \tag{2.7}$$

and hence (2.6) can be written in an equivalent form

$$\int_Q \{\sum_{i=1}^m a_i(x,t,u_n,\nabla u_n) \frac{\partial W}{\partial x_i} + c_0\, u_n\, W + f_n(x,t,u_n,\nabla u_n) W - u_n\, W_t\}\, dx\, dt = 0. \tag{2.8}$$

It follows from a priori estimate in $V^{1,0}(\nu\,\psi, Q)$ that for any positive integer n:

$$||u_n||_{1,0} \leq M. \tag{2.9}$$

With respect to (2.7) and (2.9) there exists a subsequence $\{u_n\}$ (denoted again by $\{u_n\}$) such that $\{u_n\}$ converges weakly in $V^{1,0}(\nu\psi; Q)$ and weakly* in $L^\infty(Q)$ to a function $u \in V^{1,0}(\nu\psi; Q) \cap L^\infty(Q)$ and $\operatorname*{ess\,sup}_Q |u| < L$.

We shall prove that this limit element $u \in V^{1,0}(\nu\,\psi; Q) \cap L^\infty(Q)$ is the solution of problem (1).

In order to pass to limit in (2.8) for $n \to +\infty$ we need the following auxiliary assertions:

Assertion 4

$$\lim_{n \to \infty} \int_Q \nu(x)\, \psi(t)\, |\nabla u_n(x,t) - \nabla\, u(x,t)|^2\, dx\, dt = 0.$$

Assertion 5

$$\lim_{n \to \infty} \int_Q \sum_{i=1}^m |f_n(x,t,u_n,\nabla u_n) - f(x,t,u,\nabla u)|\, dx\, dt = 0.$$

Let us suppose for a moment that the assertions are true.

Then we can pass to limit in (2.8) for n goes to $+\infty$ we obtain that

$$\int_Q \{\sum_{i=1}^{m} a_i(x,t,u,\nabla u)\frac{\partial w}{\partial x_i} + c_0\, u\, w + f(x,t,u,\nabla u)\, w - u\, w_t\}\, dx\, dt = 0$$

holds for any $w \in V_T^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$, i.e. $u \in V^{1,0}(\nu\psi; Q) \cap L^\infty(Q)$ is the solution of problem (1).

Hence it remains to prove the assertion 4-5.

Proof of assertion 4.

To this aim we shall prove that the sequence $\{u_n\}$ converges to u in $L^2(Q)$.

Let Π be rectangle such that $\Pi \subset \Omega$, $\overset{\circ}{\Pi}$ is its interior. Let τ be a positive number less than T.

We have

$$\int_\tau^T dt \int_{\overset{\circ}{\Pi}} \{u_n(x,t) + \sum_{i=1}^{m} |\frac{\partial u_n}{\partial x_i}|\}\, dx \le$$

$$\le \sqrt{\operatorname{meas} Q}\,(\int_Q |u_n|^2\, dx\, dt)^{1/2} + \sqrt{\frac{mT}{\psi(\tau)}}(\int_{\overset{\circ}{\Pi}} \frac{1}{\nu}\, dx)^{1/2}(\int_Q \nu\,\psi\, |\nabla u_n|^2\, dx\, dt)^{1/2}.$$

On the other hand if we take into the consideration (2.9), the norm of u_n in $L^1(\tau, T; W^{1,1}(\overset{\circ}{\Pi}))$ can be estimated by a constant dipendent of Π and τ but indipendent of n. Due to a preliminary estimate of the time derivative we can say the same about the norm in $L^1(\tau, T; W^{-1,1}(\overset{\circ}{\Pi}))$ of the derivatives with respect to t of the restriction of u_n on $\overset{\circ}{\Pi} \times]0,T[$. By the results of <u>*SIMON*</u> mentioned above the sequence $\{u_n\}$ has a subsequence which converges a.e. in $\overset{\circ}{\Pi} \times]0,T[$. We use also that the imbedding of $W^{1,1}(\overset{\circ}{\Pi})$ in $L^1(\overset{\circ}{\Pi})$ is compact, see <u>R.*A.Adams*</u>, Sobolev Spaces, *Academic Press, New York (1975)*. On the other hand Ω can be described as a sequence of rectangles and hence $\{u_n\}$ has a subsequence which converges a.e. in Q and by (2.7) in $L^2(Q)$. Finaly it is easy to prove that $\{u_n\}$ converges to u in $L^2(Q)$.

We shall apply lemma (2.1).

It remains to verify the assumption

$$\lim_{n\to\infty} \int_Q \sum_{i=1}^{m} [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)]\frac{\partial(u_n - u)}{\partial x_i}\, dx\, dt = 0.$$

If we denote by ω the limsup of the sequence on the left hand side we can substract a subsequence (denoted again) $\{u_n\}$ which converges a.e. in Q such that

$$\lim_{n\to\infty} \int_Q \sum_{i=1}^{m} [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)]\frac{\partial(u_n - u)}{\partial x_i}\, dx\, dt = \omega.$$

It follows from (1.3) that $\omega \ge 0$. Using a suitable test functions, the hypotheses and the properties of the sequence $\{u_n\}$ we obtain $\omega \le 0$.

Hence $\omega = 0$.

Existence result concerning the problem (2).

Theorem 1. *Let hypotheses (1.1),...., (1.11) be satisfied. Then the problem (2) has at least one solution. We shall denote by c_0 the number $c_1 + 1$.*

Put in $Q \times R^{m+1}$, $i = 1, 2, ..., m$

$$A_i(x,t,u,p) = a_i(x,t,e^{c_0 t}u, e^{c_0 t}p)\, e^{-c_0 t},$$

$$F(x,t,u,p) = f(x,t,e^{c_0 t}\,u, e^{c_0 t}\,p)\,e^{-c_0 t}.$$

It follows from precedent result thet there exists $U \in V^{1,0}(\nu\,\psi; Q) \cap L^\infty(Q)$ *such that there*

$$\int_Q \{\sum_{i=1}^{m} A_i(x,t,U,\nabla U)\frac{\partial W}{\partial x_i} + c_0\,U\,W + F(x,t,U,\nabla U)\,W - U\,W_t\}\,dx\,dt = 0 \tag{2.10}$$

for any $W \in V_T^{1,1}(\nu\,\psi, Q) \cap L^\infty(Q)$.
Let us take $e^{c_0 t}\,W$ *as a function test in (2.10).*
We get

$$\int_Q \{\sum_{i=1}^{m} a_i(x,t,e^{c_0 t}U, e^{c_0 t}\nabla U)\frac{\partial W}{\partial x_i} + f(x,t,e^{c_0 t}U, e^{c_0 t}\nabla U)W - e^{c_0 t}U W_t\}dxdt = 0.$$

Hence the function $e^{c_0 t}\,U$ *is a solution of the problem (2).*